Sequences

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Sequences

Problem 1.

(a) How many numbers are in the sequence

$$15,\ 16,\ 17,\ \ldots,\ 190,\ 191\ ?$$

(b) How many numbers are in the sequence

$$22,\ 25,\ 28,\ 31,\ \ldots,\ 160,\ 163\ ?$$

Arithmetic Progression

Solution. To answer the above question in a more general framework we need the following definition:

Definition. An **arithmetic progression** or **arithmetic sequence** is a sequence of numbers such that the difference of any two successive members of the sequence is a constant. This difference between any successive terms is called the **ratio** of the arithmetic progression.

Arithmetic Progression

For instance, the sequence

$$15,\ 16,\ 17,\ \ldots,\ 190,\ 191$$

is an arithmetic progression with ratio 1.
To find the number of the terms in an arithmetic progression we use the formula

$$\frac{\text{last term} - \text{first term}}{\text{ratio}} + 1$$

In our case the total number of terms is

$$\frac{191-15}{1} = 176 + 1 = 177 \quad \text{terms}$$

Arithmetic Progression

For the second example, the sequence

$$22,\ 25,\ 28,\ 31,\ \ldots,\ 160,\ 163$$

is an arithmetic progression with ratio 3 so the number of terms would be

$$\frac{163-22}{3} = 47 + 1 = 48$$

Arithmetic Progression

Let

$$a_1,\ a_2,\ a_3,\ \dots,\ a_n, \dots$$

be an arithmetic progression with n terms and having the ratio r.

From the above formula we find

$$\frac{a_n - a_1}{r} + 1 = n$$

Hence

$$a_n = a_1 + r(n-1)$$

Sum of terms in Arithmetic Progression

Another important formula concerns the sum of terms in an arithmetic progression

$$a_1 + a_2 + \cdots + a_n = \frac{n(a_1 + a_n)}{2}$$

In particular we have

(a) $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$

(b) $1 + 3 + 5 + \cdots + (2n - 1) = n^2$

Sums of Series

Other useful formulae are as follows

(c) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(d) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$

Sums of Series

Problem 2. For any positive integer n find the sum

$$S_n = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1)$$

Solution. Remark that

$$\begin{aligned}
S_n &= 1(1+1) + 2(2+1) + 3(3+1) + \cdots + n(n+1) \\
&= (1^2+1) + (2^2+2) + (3^2+3) + \cdots + (n^2+n) \\
&= (1^2+2^2+3^3+\cdots+n^2) + (1+2+3+\cdots+n) \\
&= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \\
&= \frac{n(n+1)}{2}\left[\frac{2n+1}{3}+1\right] \\
&= \frac{n(n+1)}{2}\frac{2n+4}{3} \\
&= \frac{n(n+1)(n+2)}{3}
\end{aligned}$$

Sums of Series

In the similar way one can compute

$$1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \cdots + (2n-1)(2n+1)$$

Problem 3. For any positive integer n find the sum

$$S_n = 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \cdots + n(n+1)(n+2)$$

Solution. The general term in the above sum is

$$k(k+1)(k+2)$$

where $k = 1, 2, 3, \ldots, n$
Remark that

$$k(k+1)(k+2) = k(k^2 + 3k + 2) = k^3 + 3k^2 + 2k$$

Sums of Series

so

$$
\begin{aligned}
S_n &= (1^3 + 3 \cdot 1^2 + 2 \cdot 1) + (2^3 + 3 \cdot 2^2 + 2 \cdot 2) + \cdots + (n^3 + 3 \cdot n^2 \\
&= (1^3 + 2^3 + \cdots + n^3) + 3(1^2 + 2^2 + \cdots + n^2) + 2(1 + 2 + \ldots \\
&= \frac{n^2(n+1)^2}{4} + 3\frac{n(n+1)(2n+1)}{6} + 2\frac{n(n+1)}{2} \\
&= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + (2n+1) + 2\right] \\
&= \frac{n(n+1)}{2}\frac{n^2+5n+6}{2} \\
&= \frac{n(n+1)(n+2)(n+3)}{4}
\end{aligned}
$$

Triangular Numbers

Problem 4. Each of the numbers

$$1 = 1, \quad 3 = 1 + 2, \quad 6 = 1 + 2 + 3, \quad 10 = 1 + 2 + 3 + 4$$

represent the number of balls that can be arranged evenly in an equilateral triangle.
This led the ancient Greeks to call a number **triangular** if it is the sum of consecutive integers beginning with 1.
Prove the following facts about triangular numbers:

(a) If n is a triangular number then $8n + 1$ is a perfect square (Plutarch, circa 100 AD)

(b) The sum of any two successive triangular numbers is a perfect square (Nicomachus, circa 100 AD)

(b) If n is a triangular number so are the numbers $9n + 1$ and $25n + 3$ (Euler, 1775)

Triangular Numbers

Solution. Remark first that n is a triangular number if there exists a positive integer k such that

$$n = 1 + 2 + 3 + \cdots + k$$

that is,

$$n = \frac{k(k+1)}{2}$$

(a) If $n = \frac{k(k+1)}{2}$ then

$$8n + 1 = 4k(k+1) + 1 = 4k^2 + 4k + 1 = (2k+1)^2$$

Triangular Numbers

(b) Let n and m be two consecutive triangular numbers. Then, there exists $k \geq 1$ such that

$$n = \frac{k(k+1)}{2} \quad \text{and} \quad m = \frac{(k+1)(k+2)}{2}$$

Then

$$n+m = \frac{k(k+1)}{2} + \frac{(k+1)(k+2)}{2} = \frac{k(k+1) + (k+1)(k+2)}{2}$$

$$n+m = \frac{(k+1)(2k+2)}{2} = (k+1)^2$$

Triangular Numbers

Problem 5. Let t_n be the nth triangular number, that is

$$t_1 = 1, \quad t_2 = 3, \quad t_3 = 6, \quad t_4 = 10, \ldots$$

Prove the formula

$$t_1 + t_2 + \cdots + t_n = \frac{n(n+1)(n+2)}{6}$$

Triangular Numbers

Solution.
We have

$$t_n = \frac{n(n+1)}{2} = \frac{n^2+n}{2}.$$

Therefore,

$$\begin{aligned} t_1 + t_2 + \cdots + t_n &= \frac{1^2+1}{2} + \frac{2^2+2}{2} + \frac{3^2+3}{2} + \cdots + \frac{n^2+n}{2} \\ &= \frac{1^2+2^2+3^2+\cdots+n^2}{2} \\ &+ \frac{1+2+3+\cdots+n}{2} \\ &= \frac{1}{2}\big[(1^2+2^2+3^2+\cdots+n^2) \\ &+ (1+2+\cdots+n)\big] \\ &= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right] \end{aligned}$$

Triangular Numbers

$$= \frac{1}{2}\,\frac{n(n+1)}{2}\left[\frac{2n+1}{3}+1\right] \quad = \frac{1}{2}\,\frac{n(n+1)}{2}\,\frac{2n+4}{3}$$

$$= \frac{n(n+1)(2n+4)}{12}$$

$$= \frac{n(n+1)(n+2)}{6}$$

Arithmetic Progressions with Perfect Squares

Problem 6. Prove that if an infinite arithmetic progression of positive integers contains a perfect square, then it contains an infinite number of perfect squares.

Solution. Let

$$a_1 < a_2 < \cdots < a_n < a_{n+1} < \dots$$

be an infinite arithmetic progression containing a perfect square, say a^2.

Arithmetic Progressions with Perfect Squares

Denote by r its ratio. Then, the numbers

$$a^2,\ a^2+r,\ a^2+2r,\ \ldots, a^2+kr$$

are terms of the above arithmetic progression, $k = 1, 2, 3, \ldots$.
In particular the number

$$a^2 + r(2a + r) = a^2 + 2ar + r^2 = (a + r)^2$$

is a perfect square and is another term of the above arithmetic progression.

Arithmetic Progressions with Perfect Squares

Thus,

$$(a+r)^2,\ (a+r)^2+r,\ \ldots,\ (a+r)^2+kr, \ldots$$

are terms of the initial arithmetic progression.
As above, it follows that

$$(a+r)^2 + r[2(a+r)+r] = (a+2r)^2$$

is a perfect square and belongs to the initial arithmetic progression.
We have obtained so far that $(a+r)^2$, $(a+2r)^2$ are terms in the progression.
Proceeding similarly we obtain that all the perfect squares

$$(a+r)^2,\ (a+2r)^2,\ \ldots,\ (a+100r)^2, \ldots$$

are terms in the initial arithmetic progression.

Arithmetic Progressions of Perfect Squares

Problem 7. Prove that there are no arithmetic progressions of positive integers whose terms are all perfect squares.

Solution. Assume by **contradiction** that there exists positive integers

$$a_1 < a_2 < \dots < a_n < a_{n+1} < \dots$$

such that

$$a_1^2 < a_2^2 < \dots < a_n^2 < a_{n+1}^2 < \dots$$

is an arithmetic progression.

Then, the ratio of it would be

$$r = a_2^2 - a_1^2 = a_3^2 - a_2^2 = \dots = a_n^2 - a_{n-1}^2 = a_{n+1}^2 - a_n^2 = \dots$$

It follows that

$$(a_n - a_{n-1})(a_n + a_{n-1}) = (a_{n+1} - a_n)(a_{n+1} + a_n), \quad n = 2, 3, 4, \dots$$

Arithmetic Progressions of Perfect Squares

Since $a_{n-1} < a_n < a_{n+1}$ we have $a_{n+1} + a_n > a_n + a_{n-1}$ so the above equality yields

$$a_2 - a_1 > a_3 - a_2 > a_4 - a_3 > \cdots > a_n - a_{n-1} > \cdots > 0$$

which is clearly impossible.